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The Da Rios system under a geometric constraint: the Gilbarg problem

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Abstract

A classical problem in hydrodynamics originally posed by Gilbarg has been recently reduced to that of solving a solitonic Heisenberg spin equation subject to a geometric constraint. Here, this reformulation is shown to lead to a class of solutions of the Gilbarg problem corresponding to travelling wave solutions of a system derived by Da Rios in 1906.

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1. Introduction

A long-standing problem in hydrodynamics posed by Gilbarg [1] and subsequently investigated by Prim [2], Howard [3], Wasserman [4] and Marris [5] has recently been shown to be encapsulated in a nonlinear system consisting of an integrable Heisenberg spin

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equation subject to a geometric constraint [6,7]. This Heisenberg spin equation is equivalent to the celebrated nonlinear Schrödinger (NLS) equation which, in turn, is a consequence of the classical Da Rios system which was originally set down in 1906 in connection with the spatial evolution of an isolated vortex filament in an unbounded inviscid liquid [8]. The hydrodynamics problem treated in [1–7], in fact, represents a generalisation of a well-known problem posed and resolved by Hamel [9]. An alternative demonstration of what has come to be known as Hamel's theorem has been given in [10] via a geometric formalism originally introduced by Marris and Passman [11] in a kinematic study of hydrodynamics. This formalism has been exploited in magnetohydrodynamics [12] and recently within the context of the geometry of soliton theory [13,14]. Here, it is used to address what we term the Gilbarg problem which seeks to delimit steady hydrodynamic motions for which the speed of the fluid flow is constant along streamlines. In view of the continuity equation, this condition is equivalent to the purely geometric constraint $\operatorname{div} \boldsymbol{t} = 0$, where $\boldsymbol{t}$ is the unit tangent to a generic streamline. It is established that for such motions, two important geometric constraints must apply on the abnormality $\Omega = \boldsymbol{t} \cdot \operatorname{curl} \boldsymbol{t}$. Remarkably, it is demonstrated that these constraints encode the 'travelling wave' symmetry reduction of the Da Rios system if Ω is assumed to be constant on the constant pressure surfaces. This result encapsulates Hamel's theorem corresponding to $\Omega = 0$. The motions of Gilbarg type which are compatible with the travelling wave reduction of the Da Rios system have recently been delimited in [15].

It is of interest to remark that the geometric concept of abnormality and, indeed, constant abnormality plays an important role in the advance made by Marris [16,17] in his investigation of Ericksen's problem to determine all deformations that can be sustained by a perfectly elastic, isotropic, incompressible body subject only to surface tractions [18]. Marris' contribution to the study of Ericksen's problem has recently been discussed in a survey on universal solutions in elasticity by Saccomandi [19]. Universal states in anti-plane shear and connections with hydrodynamics have also been discussed by Knowles [20].

An analogue of Gilbarg's problem may also be formulated in the context of magnetohydrodynamics. In that case, it has been shown that the integrable Pohlmeyer–Lund–Regge model subject to a volume-preserving constraint arises as an exact reduction of the equilibrium equations [21]. Moreover, the above-mentioned Heisenberg spin connection is retrieved in the hydrodynamic or magnetohydrostatic limit.

2. The class of hydrodynamic motions

Here, we consider the classical system of steady hydrodynamics

$$\operatorname{div} \boldsymbol{q} = 0, \qquad \rho(\boldsymbol{q} \cdot \nabla)\boldsymbol{q} + \nabla p = \mathbf{0}, \tag{2.1}$$

where $\boldsymbol{q}$ is the fluid velocity and p, ρ are the pressure and constant density, respectively. In this context, the Gilbarg problem [1] may be formulated as follows:

Under what circumstances is a flow uniquely determined by its streamline pattern?

In [2], Prim established that "any flow is unique unless it has a constant velocity magnitude along each individual streamline." Thus, up to a trivial scaling of the velocity magnitude

$q = |\boldsymbol{q}|$, the flow may be reconstructed from its streamline pattern unless

$$\boldsymbol{q} \cdot \nabla q = 0. \tag{2.2}$$

Indeed, if the latter condition holds then there exists a multiplicity of flows which exhibit the same streamline pattern since the system (2.1), (2.2) is seen to be invariant under $(p, \boldsymbol{q}) \to (P(p), \sqrt{P'(p)}\,\boldsymbol{q})$.

Gilbarg [1] and Prim [2] resolved Gilbarg's problem for planar and axisymmetric flows, respectively and showed that the streamlines are necessarily concentric circles or straight lines. Wasserman [4] reformulated the governing equations in the language of tensor calculus and reduced the rotationally symmetric case to a pair of ordinary differential equations. The latter has been recently shown to be solvable in terms of complete elliptic integrals [15,22]. In particular, there exist configurations of nested constant pressure tori spanned by the geodesic streamlines. However, Prim [2] states that for general spatial flows, the geometric implications of his theorem are unknown. Marris [5] derived necessary constraints on the geometry of the admissible flows. These 'constant speed flows' were the subject of the thesis of Howard [3]. Their analysis may be extended to Prim gases [4] and, as demonstrated below, to compressible fluids with arbitrary state law. In [6], a remarkable link was established between the hydrodynamic system (2.1), (2.2) and the integrable Heisenberg spin equation. It has subsequently been shown [7] that the hydrodynamics system in question is completely encapsulated in the Heisenberg spin equation subject to a geometric constraint. The result may be summarised as follows.

Theorem 1. *Steady hydrodynamic motions with $\boldsymbol{q} \cdot \nabla q = 0$ are governed by the purely geometric system*

$$\frac{\partial \boldsymbol{t}}{\partial b} = \boldsymbol{t} \times \frac{\partial^2 \boldsymbol{t}}{\partial s^2}, \quad \operatorname{div} \boldsymbol{t} = 0, \tag{2.3}$$

where $\boldsymbol{q} = q\boldsymbol{t}$ while s denotes arc length along the streamlines and b parametrises their orthogonal trajectories on the constant pressure surfaces. The streamlines and the orthogonal trajectories constitute geodesics and parallels respectively thereon. The velocity $\boldsymbol{q}$ and the pressure p are determined by integration of the compatible system

$$\frac{\delta q}{\delta s} = 0, \qquad \frac{\delta \ln q}{\delta b} = -\frac{1}{2\kappa} \operatorname{div}(\kappa \boldsymbol{b}),$$
$$\frac{\delta p}{\delta s} = 0, \qquad \frac{\delta p}{\delta n} = -\rho q^2 \kappa, \quad \frac{\delta p}{\delta b} = 0, \tag{2.4}$$

where

$$\frac{\delta}{\delta s} = \boldsymbol{t} \cdot \nabla, \qquad \frac{\delta}{\delta n} = \boldsymbol{n} \cdot \nabla, \qquad \frac{\delta}{\delta b} = \boldsymbol{b} \cdot \nabla \tag{2.5}$$

designate the directional derivatives, in turn, in the tangential, principal normal and binormal directions to the streamlines.

It is with the Heisenberg spin equation $(2.3)_1$ subject to the constraint $(2.3)_2$ that we shall be concerned in the present paper.

The above theorem and the subsequent analysis apply 'mutatis mutandis' to steady motions of an inviscid and thermally nonconducting compressible fluid. In that context, the governing equations are

$$\operatorname{div}(\rho \boldsymbol{v}) = 0, \qquad \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = \boldsymbol{0}, \qquad \boldsymbol{v}\cdot\nabla\eta = 0 \tag{2.6}$$

augmented by an equation of state of the form

$$\rho = \rho(p, \eta), \tag{2.7}$$

wherein $\boldsymbol{v}$ is the steady fluid velocity while p, ρ and η denote the pressure, density and specific entropy respectively. If we assume that the streamlines lie on the constant pressure surfaces, that is

$$\boldsymbol{v}\cdot\nabla p = 0, \tag{2.8}$$

then the equation of motion $(2.6)_2$, the equation of state (2.7) and $(2.6)_3$ imply that

$$\boldsymbol{v}\cdot\nabla v = 0, \qquad \boldsymbol{v}\cdot\nabla\rho = 0, \tag{2.9}$$

where $\boldsymbol{v} = v\boldsymbol{t}$. Accordingly, on introduction of the canonical variable

$$\boldsymbol{q} = q\boldsymbol{t} = \sqrt{\rho}\,\boldsymbol{v}, \tag{2.10}$$

the governing equations reduce to the system $(2.1)_{\rho=1}$, (2.2), that is

$$\operatorname{div}\boldsymbol{q} = 0, \qquad (\boldsymbol{q}\cdot\nabla)\boldsymbol{q} + \nabla p = \boldsymbol{0}, \qquad \boldsymbol{q}\cdot\nabla q = 0, \tag{2.11}$$

along with the isentropic condition

$$\boldsymbol{q}\cdot\nabla\eta = 0. \tag{2.12}$$

It is important to note that the particular motions considered here do not impose any constraints on the equation of state.

3. Geometric preliminaries

Here, the orthonormal basis $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ consisting of the unit tangent $\boldsymbol{t}$, principal normal $\boldsymbol{n}$ and binormal $\boldsymbol{b}$ to the streamlines is adopted. The directional derivatives along these unit vectors are defined by (2.5). Following [23], we introduce the notation

$$\theta_{ns} = \boldsymbol{n}\cdot\frac{\delta\boldsymbol{t}}{\delta n}, \qquad \theta_{bs} = \boldsymbol{b}\cdot\frac{\delta\boldsymbol{t}}{\delta b} \tag{3.1}$$

and

$$\Omega = \boldsymbol{t}\cdot\operatorname{curl}\boldsymbol{t}, \qquad \Omega_n = \boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}, \qquad \Omega_b = \boldsymbol{b}\cdot\operatorname{curl}\boldsymbol{b}. \tag{3.2}$$

It may be established that

$$\operatorname{div}\boldsymbol{t} = \theta_{ns} + \theta_{bs}, \qquad \Omega_b = \Omega - \Omega_n - 2\tau, \qquad \operatorname{curl}\boldsymbol{t} = \Omega\boldsymbol{t} + \kappa\boldsymbol{b}. \tag{3.3}$$

The latter relation was originally obtained by Masotti [24] and was later rediscovered independently by Emde [25] and Bjørgum [23]. Application of the identity $\operatorname{curl}\operatorname{grad}\phi = \boldsymbol{0}$

to a test function ϕ now produces the commutator relations

$$\frac{\delta^2}{\delta n\delta b}-\frac{\delta^2}{\delta b\delta n}=-\Omega\frac{\delta}{\delta s}+\operatorname{div}\boldsymbol{b}\frac{\delta}{\delta n}-(\kappa+\operatorname{div}\boldsymbol{n})\frac{\delta}{\delta b},$$
$$\frac{\delta^2}{\delta b\delta s}-\frac{\delta^2}{\delta s\delta b}=-\Omega_n\frac{\delta}{\delta n}+\theta_{bs}\frac{\delta}{\delta b},$$
$$\frac{\delta^2}{\delta s\delta n}-\frac{\delta^2}{\delta n\delta s}=-\kappa\frac{\delta}{\delta s}-\theta_{ns}\frac{\delta}{\delta n}-\Omega_b\frac{\delta}{\delta b}. \tag{3.4}$$

The directional derivatives (2.5) of the orthonormal triad $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ may be shown to be given by [26]

$$\frac{\delta}{\delta s}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix}=\begin{pmatrix}0&\kappa&0\\ -\kappa&0&\tau\\ 0&-\tau&0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix},$$
$$\frac{\delta}{\delta n}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix}=\begin{pmatrix}0&\theta_{ns}&\Omega_b+\tau\\ -\theta_{ns}&0&-\operatorname{div}\boldsymbol{b}\\ -(\Omega_b+\tau)&\operatorname{div}\boldsymbol{b}&0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix},$$
$$\frac{\delta}{\delta b}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix}=\begin{pmatrix}0&-(\Omega_n+\tau)&\theta_{bs}\\ \Omega_n+\tau&0&\kappa+\operatorname{div}\boldsymbol{n}\\ -\theta_{bs}&-(\kappa+\operatorname{div}\boldsymbol{n})&0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix}. \tag{3.5}$$

On use of the commutator relations (3.4), the compatibility of the above system is seen to impose the following set of nine conditions on the eight geometric quantities κ, τ, Ω, Ω_n, $\operatorname{div}\boldsymbol{n}$, $\operatorname{div}\boldsymbol{b}$, θ_{ns} and θ_{bs} [26]:

$$\frac{\delta\theta_{ns}}{\delta b}+\frac{\delta}{\delta n}(\tau+\Omega_n)=(\kappa+\operatorname{div}\boldsymbol{n})(\Omega-2\Omega_n-2\tau)+\operatorname{div}\boldsymbol{b}\,(\theta_{bs}-\theta_{ns})+\Omega\kappa,$$
$$\frac{\delta}{\delta b}(\tau+\Omega_n-\Omega)+\frac{\delta\theta_{bs}}{\delta n}=(\kappa+\operatorname{div}\boldsymbol{n})(\theta_{ns}-\theta_{bs})+\operatorname{div}\boldsymbol{b}\,(\Omega-2\Omega_n-2\tau),$$
$$\frac{\delta}{\delta b}\operatorname{div}\boldsymbol{b}+\frac{\delta}{\delta n}(\kappa+\operatorname{div}\boldsymbol{n})=(\tau+\Omega_n)(\tau+\Omega_n-\Omega)-\theta_{ns}\theta_{bs}-\tau\Omega-(\operatorname{div}\boldsymbol{b})^2 -(\kappa+\operatorname{div}\boldsymbol{n})^2$$
$$\frac{\delta}{\delta s}(\tau+\Omega_n)+\frac{\delta\kappa}{\delta b}=-\Omega_n\theta_{ns}-(2\tau+\Omega_n)\theta_{bs},$$
$$\frac{\delta\theta_{bs}}{\delta s}=-\theta_{bs}^2+\kappa(\kappa+\operatorname{div}\boldsymbol{n})-\Omega_n(\tau+\Omega_n-\Omega)+\tau(\tau+\Omega_n),$$
$$\frac{\delta}{\delta s}(\kappa+\operatorname{div}\boldsymbol{n})-\frac{\delta\tau}{\delta b}=-\Omega_n\operatorname{div}\boldsymbol{b}-\theta_{bs}(2\kappa+\operatorname{div}\boldsymbol{n}),$$
$$\frac{\delta\kappa}{\delta n}-\frac{\delta\theta_{ns}}{\delta s}=\kappa^2+\theta_{ns}^2+(\tau+\Omega_n)(3\tau+\Omega_n)-\Omega(2\tau+\Omega_n),$$

$$\frac{\delta}{\delta s}(\tau + \Omega_n - \Omega) = -\theta_{ns}(\Omega_n - \Omega) + \kappa \operatorname{div} \boldsymbol{b} + \theta_{bs}(-2\tau - \Omega_n + \Omega),$$
$$\frac{\delta \tau}{\delta n} + \frac{\delta}{\delta s} \operatorname{div} \boldsymbol{b} = -\kappa(\Omega_n - \Omega) - \theta_{ns} \operatorname{div} \boldsymbol{b} + (\kappa + \operatorname{div} \boldsymbol{n})(-2\tau - \Omega_n + \Omega). \tag{3.6}$$

4. Intrinsic decomposition of the hydrodynamic system

The hydrodynamic system (2.1), on intrinsic decomposition, yields

$$\frac{\delta q}{\delta s} + q \operatorname{div} \boldsymbol{t} = 0, \qquad \frac{\delta p}{\delta s} = \rho q^2 \operatorname{div} \boldsymbol{t}, \qquad \frac{\delta p}{\delta n} = -\rho q^2 \kappa, \qquad \frac{\delta p}{\delta b} = 0. \tag{4.1}$$

Application of the commutator relations (3.4) shows that the compatibility conditions on the pressure distribution yield

$$2\left(\frac{\delta \ln q}{\delta n}\right) \operatorname{div} \boldsymbol{t} = -\frac{\delta \kappa}{\delta s} + \theta_{bs}\kappa - \frac{\delta}{\delta n} \operatorname{div} \boldsymbol{t} + 2\kappa \operatorname{div} \boldsymbol{t},$$
$$2\left(\frac{\delta \ln q}{\delta b}\right) \operatorname{div} \boldsymbol{t} = \kappa \Omega_n - \frac{\delta}{\delta b} \operatorname{div} \boldsymbol{t},$$
$$2\kappa \left(\frac{\delta \ln q}{\delta b}\right) = -\operatorname{div}(\kappa \boldsymbol{b}) - \Omega \operatorname{div} \boldsymbol{t}. \tag{4.2}$$

Under the constraint (2.2), namely

$$\frac{\delta q}{\delta s} = 0, \tag{4.3}$$

the system $(4.1)_1$, (4.2) reduces to the purely geometric set of equations

$$\operatorname{div} \boldsymbol{t} = 0,\ \Omega_n = 0, \qquad \frac{\delta \kappa}{\delta s} = \theta_{bs}\kappa \tag{4.4}$$

together with

$$\frac{\delta q}{\delta b} = -\frac{q}{2\kappa} \operatorname{div}(\kappa \boldsymbol{b}) \tag{4.5}$$

provided that $\kappa \neq 0$. Importantly, the variation of the speed q in the direction of the principal normal remains arbitrary. Accordingly, as discussed in Section 2, constant speed flows are not completely determined by their streamline geometry.

On application of the commutator relation $(3.4)_2$, the compatibility of the relations (4.3) and (4.5) is readily shown to lead to the requirement

$$\frac{\delta}{\delta s} \operatorname{div}(\kappa \boldsymbol{b}) = 0. \tag{4.6}$$

However, the latter is a consequence of the compatibility conditions (3.6). Indeed, elimination of $\delta\tau/\delta n$ between $(3.6)_1$ and $(3.6)_9$ results in

$$\frac{\delta}{\delta s}\operatorname{div} \boldsymbol{b} + \operatorname{div}(\theta_{bs}\boldsymbol{b}) = 0. \tag{4.7}$$

Insertion of θ_{bs} as given by (4.4)$_3$ and use of the commutator relation (3.4)$_2$ then produce (4.6).

5. Constraints on the abnormality $\boldsymbol{\Omega}$

The geometric relations (3.6) and (4.4) encapsulate two important constraints on the abnormality Ω. Firstly, elimination of $\delta\tau/\delta s$ from (3.6)$_{4,8}$ leads to

$$\frac{\delta\Omega}{\delta s} + \Omega\,\mathrm{div}\,\boldsymbol{t} + \mathrm{div}\,(\kappa\boldsymbol{b}) = 0, \tag{5.1}$$

which is nothing but the identity $\mathrm{div}\,\mathrm{curl}\,\boldsymbol{t} = 0$, where $\mathrm{curl}\,\boldsymbol{t}$ is given by the Masotti–Emde–Bjørgum relation (3.3)$_3$. In view of (4.4)$_1$ and (4.6), the relation (5.1) implies that

$$\frac{\delta^2\Omega}{\delta s^2} = 0. \tag{5.2}$$

Secondly, addition of the relations (3.6)$_{2,6}$ produces

$$\frac{\delta\Omega}{\delta b} = \frac{\delta\theta_{bs}}{\delta n} + \frac{\delta}{\delta s}(\kappa + \mathrm{div}\,\boldsymbol{n}) + 3\theta_{bs}(\kappa + \mathrm{div}\,\boldsymbol{n}) + \theta_{bs}\kappa + (2\tau - \Omega)\mathrm{div}\,\boldsymbol{b}. \tag{5.3}$$

The commutator relation (3.4)$_3$ yields

$$\frac{\delta\theta_{bs}}{\delta n} = \frac{\delta^2 \ln\kappa}{\delta n\delta s} = \frac{\delta^2 \ln\kappa}{\delta s\delta n} + \kappa\frac{\delta \ln\kappa}{\delta s} - \theta_{bs}\frac{\delta \ln\kappa}{\delta n} + \Omega_b\frac{\delta \ln\kappa}{\delta b}, \tag{5.4}$$

wherein, in view of (3.6)$_{4,7}$, the directional derivatives $\delta\kappa/\delta b$ and $\delta\kappa/\delta n$ may be replaced by

$$\frac{\delta\kappa}{\delta b} = -\frac{\delta\tau}{\delta s} - 2\tau\theta_{bs}, \qquad \frac{\delta\kappa}{\delta n} = -\frac{\delta\theta_{bs}}{\delta s} + \kappa^2 + \theta_{bs}^2 + 3\tau^2 - 2\tau\Omega. \tag{5.5}$$

Moreover, on solving (3.6)$_5$ and (3.6)$_8$ for $\kappa + \mathrm{div}\,\boldsymbol{n}$ and $\mathrm{div}\,\boldsymbol{b}$, respectively, one obtains

$$\kappa + \mathrm{div}\,\boldsymbol{n} = \frac{1}{\kappa}\left(\frac{\delta\theta_{bs}}{\delta s} + \theta_{bs}^2 - \tau^2\right), \qquad \mathrm{div}\,\boldsymbol{b} = \frac{1}{\kappa}\left(\frac{\delta\tau}{\delta s} - \frac{\delta\Omega}{\delta s} + 2\tau\theta_{bs}\right). \tag{5.6}$$

Reduction of (5.3) by means of (5.4)–(5.6) now yields

$$\kappa\frac{\delta\Omega}{\delta b} = \frac{\delta}{\delta s}\left(4\tau^2 + 4\theta_{bs}^2 + \kappa^2 - 4\tau\Omega + \frac{\Omega^2}{2}\right). \tag{5.7}$$

6. The geometry of the constant pressure surfaces: a constrained Da Rios system

The equation of motion (2.1)$_2$ together with the constraint (2.2) shows that

$$\nabla p = -\rho q^2\kappa\boldsymbol{n}, \tag{6.1}$$

whence the principal normal to the streamlines is parallel to the normal to the constant pressure surfaces. Accordingly, the streamlines, namely the s-lines, are geodesics on the constant pressure surfaces while their orthogonal trajectories, the b-lines, are necessarily parallels thereon. Indeed, the vanishing abnormality condition Ω_n provides a necessary and sufficient condition for surfaces to exist which contain the trajectories of the $\boldsymbol{t}$ and $\boldsymbol{b}$ vector fields. If these s-lines and b-lines are now taken as parametric curves on the constant pressure surfaces then their metric adopts the geodesic form

$$I = \mathrm{d}s^2 + h^2\,\mathrm{d}b^2, \tag{6.2}$$

where

$$\nabla_{p=\mathrm{const.}} = \boldsymbol{t}\frac{\delta}{\delta s} + \boldsymbol{b}\frac{\delta}{\delta b} = \boldsymbol{t}\frac{\partial}{\partial s} + \frac{\boldsymbol{b}}{h}\frac{\partial}{\partial b}, \tag{6.3}$$

and

$$\theta_{bs} = \frac{\partial \ln h}{\partial s}. \tag{6.4}$$

The relations $(3.6)_{4,5,6}$ are nothing but the Gauß–Mainardi–Codazzi equations for the constant pressure surfaces and become

$$\frac{\partial\kappa}{\partial b} = -\frac{1}{h}\frac{\partial}{\partial s}(h^2\tau), \qquad \frac{\partial\tau}{\partial b} = \frac{\partial}{\partial s}[h(\kappa + \operatorname{div}\boldsymbol{n})] + \kappa\frac{\partial h}{\partial s},$$
$$\frac{\partial^2 h}{\partial s^2} = [\kappa(\kappa + \operatorname{div}\boldsymbol{n}) + \tau^2]h. \tag{6.5}$$

However, the relation $(4.4)_3$ shows that

$$h = \lambda\kappa, \tag{6.6}$$

where $\delta\lambda/\delta s = 0$. On transforming $\lambda b \to b$, the directional derivatives in (6.3) become

$$\frac{\delta}{\delta s} = \frac{\partial}{\partial s}, \qquad \frac{\delta}{\delta b} = \frac{1}{\kappa}\frac{\partial}{\partial b} \tag{6.7}$$

so that, on elimination of $\kappa + \operatorname{div}\boldsymbol{n}$, Eqs. (5.2), (5.7) and (6.5) are seen to represent a system of two-dimensional differential equations which prevail on any individual constant pressure surface. Remarkably, the pair $(6.5)_{1,2}$ constitutes the celebrated Da Rios system [8,27,28].[1] This gives rise to the following key observation.

Theorem 2. *In steady hydrodynamic motions with* $\delta q/\delta s = 0$, *the curvature and torsion of the streamlines obey the Da Rios system*

$$\frac{\partial\kappa}{\partial b} = -2\frac{\partial\kappa}{\partial s}\tau - \kappa\frac{\partial\tau}{\partial s}, \qquad \frac{\partial\tau}{\partial b} = \frac{\partial}{\partial s}\left(\frac{1}{\kappa}\frac{\partial^2\kappa}{\partial s^2} + \frac{\kappa^2}{2} - \tau^2\right), \tag{6.8}$$

[1] L.S. Da Rios engaged in a long-standing investigation of the motion of three-dimensional vortex filaments starting in 1906 under the guidance of T. Levi-Civita at the University of Padua. He originated the now celebrated localised induction approximation (LIA) to be re-discovered more than 50 years later. An excellent account of the contributions of Da Rios and Levi-Civita to the study of vortex dynamics has been given by Ricca [29].

where s denotes arc length along the streamlines and b parametrises their orthogonal trajectories on the constant pressure surfaces. The abnormality Ω is subject to the two necessary constraints (5.2) and (5.7), that is

$$\frac{\partial^2 \Omega}{\partial s^2} = 0, \qquad \frac{\partial \Omega}{\partial b} = \frac{\partial}{\partial s}\left[4\tau^2 + 4\left(\frac{1}{\kappa}\frac{\partial \kappa}{\partial s}\right)^2 + \kappa^2 - 4\tau\Omega + \frac{\Omega^2}{2}\right]. \tag{6.9}$$

The Da Rios system is equivalent to the Heisenberg spin equation $(2.3)_1$. In general, the two conditions (6.9) on the abnormality Ω are not compatible and impose constraints on the Da Rios system (6.8). It is with these constraints that we shall be concerned in the remainder of this paper.

7. A class of solutions of the Gilbarg problem

Hamel [9], in 1937, undertook a detailed study of irrotational constant speed flows. Thus, he was concerned with the existence of solutions of the hydrodynamic system (2.1), (2.2) subject to the additional constraint

$$\operatorname{curl} \boldsymbol{q} = \boldsymbol{0}. \tag{7.1}$$

Since, in this case, $\Omega = 0$ and $\operatorname{div} \boldsymbol{t} = 0$, there exist minimal surfaces [30] which are orthogonal to the streamlines. Hamel employed the classical Weierstrass representation [31] of minimal surfaces to show that the streamlines of such flows can only be straight lines or helices mounted on concentric circular cylinders. In [10], Marris presented an alternative demonstration of Hamel's theorem. This involved establishing that the geometric systems (3.6), (4.4) subject to (7.1) are compatible if and only if $\kappa = 0$ or $\delta\kappa/\delta s = \delta\tau/\delta s = 0$. Prim [32] showed that the admissible streamline geometries of constant speed flows which are complex-lamellar, that is

$$\Omega = 0, \tag{7.2}$$

coincide with those of irrotational constant speed flows. Hamel's theorem may therefore be called upon to deduce that the streamlines of complex-lamellar constant speed flows are likewise either straight lines or circular helices.

Here, Prim's (and 'a fortiori' Hamel's) condition is relaxed. It is demanded that the abnormality Ω be constant on individual constant pressure surfaces. This is motivated by the fact that, even though the geometric meaning of the condition (7.2) is evident, from the point of Lie point symmetries, it is not an invariant condition. In this connection, it is noted that, in principle, one should be able to bring the overdetermined system (6.8), (6.9) into involutive form in the sense of Cartan [33] and Kähler [34] by adding all necessary compatibility conditions. However, it turns out that, in the case of a generic abnormality, the practical implementation of this statement by means of a computer algebra package such as RIF [35] appears to be obstructed by the high level of computational complexity.

The following theorem constitutes the key result of the present paper.

Theorem 3. *In steady hydrodynamic motions with*

$$\frac{\delta q}{\delta s} = 0, \qquad \frac{\delta \Omega}{\delta s} = 0, \qquad \frac{\delta \Omega}{\delta b} = 0, \tag{7.3}$$

the curvature κ *and torsion* τ *of the streamlines are necessarily 'travelling wave' solutions*

$$\kappa = \kappa(s + cb), \qquad \tau = \tau(s + cb) \tag{7.4}$$

of the Da Rios system (6.8). The solution of the constrained Da Rios system (6.8), (6.9) is given by

$$\kappa = \sqrt{\rho}, \qquad \tau = \frac{\Omega}{2} + \alpha + \frac{\beta}{\rho}, \tag{7.5}$$

where $c = -\Omega - 2\alpha$ *and* ρ *is the elliptic function defined by*

$$\rho'^2 + \rho^3 - 4\gamma\rho^2 + 4(\alpha\rho + \beta)^2 = 0. \tag{7.6}$$

Here, the quantities α, β, γ *and* δ *are independent of* s *and* b*. The streamlines on any individual constant pressure surface are identical up to Euclidean motions. The latter are generated by the compatible constraint*

$$\frac{\partial \boldsymbol{t}}{\partial b} = c\frac{\partial \boldsymbol{t}}{\partial s} + \boldsymbol{c} \times \boldsymbol{t} \tag{7.7}$$

on the Heisenberg spin equation $(2.3)_1$*, where the vector* $\boldsymbol{c}$ *is likewise constant on any individual constant pressure surface. The streamlines have the shape of (the centreline of) symmetric elastic rods.*

Proof. The Gilbarg problem of delimiting hydrodynamic motions subject to the constraint (2.2) has been seen to reduce to the problem of solving the solitonic Da Rios system

$$\kappa_b = -2\kappa_s\tau - \kappa\tau_s, \qquad \tau_b = \left(\frac{\kappa_{ss}}{\kappa} + \frac{\kappa^2}{2} - \tau^2\right)_s \tag{7.8}$$

subject to the constraint $\operatorname{div} \boldsymbol{t} = 0$. Here, subscripts designate partial derivatives. Our starting point is the observation that the Da Rios system and the conditions

$$\Omega_{ss} = 0, \qquad \Omega_b = \left[4\tau^2 + 4\left(\frac{\kappa_s}{\kappa}\right)^2 + \kappa^2 - 4\tau\Omega + \frac{\Omega^2}{2}\right]_s \tag{7.9}$$

on the abnormality of the $\boldsymbol{t}$ vector field admit common Lie point symmetries. Thus, the important invariance [39]

$$\partial_b \to \partial_b - 2\lambda\partial_s, \qquad \tau \to \tau + \lambda \tag{7.10}$$

of the Da Rios system (7.8) which is known to inject the constant 'spectral' parameter λ into the associated Lax representation extends to the system (7.9) if it is supplemented by

$$\Omega \to \Omega + 2\lambda. \tag{7.11}$$

Hence, if Ω is constant on individual constant pressure surfaces then it may be removed by means of the above symmetry and the constraint on the Da Rios system becomes

$$4\tau^2 + 4\left(\frac{\kappa_s}{\kappa}\right)^2 + \kappa^2 = 4\gamma(b), \tag{7.12}$$

where γ constitutes an arbitrary function of integration. However, it is emphasised that once a solution of the Da Rios system and the constraint (7.12) has been found, Ω must be retrieved since the above invariance does not necessarily extend to the complete hydrodynamic system. In particular, Ω may depend on the pressure p.

The case $\kappa_s = 0$: If the curvature κ is constant along the streamlines then the constraint (7.12) implies that the torsion τ is likewise independent of s. Accordingly, the streamlines constitute helices and the Da Rios equations (7.8) show that κ and τ are constant on each individual constant pressure surface. It is well-known [5] that these helices are mounted on concentric circular cylinders.

The case $\tau = 0$: If the streamlines are planar then the Da Rios equation $(7.8)_1$ implies that κ is independent of b. Accordingly, γ is constant and the remaining Da Rios equation $(7.8)_2$ is satisfied modulo the constraint (7.12) which may be solved in terms of elliptic functions.

The case $\gamma =$ const.: Here, we investigate under what circumstances the constraint (7.12) is compatible with the Da Rios system if γ is independent of b. In terms of

$$\rho = \kappa^2, \tag{7.13}$$

the constraint (7.12) may be written as

$$\tau = \frac{\sigma}{\rho}, \qquad \sigma^2 = \gamma\rho^2 - \frac{1}{4}\rho^3 - \frac{1}{4}\rho_s^2, \tag{7.14}$$

and the Da Rios equations become

$$\rho_b = -2(\rho\tau)_s, \qquad \tau_b = \left(\frac{1}{2}\frac{\rho_{ss}}{\rho} - \frac{1}{4}\frac{\rho_s^2}{\rho^2} + \frac{\rho}{2} - \tau^2\right)_s. \tag{7.15}$$

Insertion of τ as given by $(7.14)_1$ into $(7.15)_1$ yields

$$\rho_b = -2\sigma_s, \tag{7.16}$$

while evaluation of $(7.15)_2$ produces

$$\left(\frac{\sigma_s}{\rho_s}\right)_s = 0. \tag{7.17}$$

In the derivation of the latter, derivatives of ρ in binormal direction have been removed by means of (7.16). Accordingly, σ and ρ are related by

$$\sigma = \alpha(b)\rho + \beta(b) \tag{7.18}$$

so that (7.16) becomes

$$\rho_b = -2\alpha\rho_s. \tag{7.19}$$

The latter is compatible with (7.14)$_2$, that is

$$\rho_s^2 + \rho^3 - 4\gamma\rho^2 + 4(\alpha\rho + \beta)^2 = 0, \tag{7.20}$$

if and only if

$$\alpha_b = \beta_b = 0. \tag{7.21}$$

Thus, it has been demonstrated that the condition $\gamma =$ const. leads to travelling wave solutions of the Da Rios system which may be expressed in terms of elliptic functions by virtue of (7.20).

The case $\gamma \neq$ const.: Here, it will be shown that the constraint (7.12) is incompatible with the Da Rios system if $\gamma_b \neq 0$. As in the preceding, we introduce the quantities ρ and σ so that the relations (7.13)–(7.16) are still valid. However, evaluation of the Da Rios equation (7.15)$_2$ now yields

$$\gamma_b + (4\gamma - \rho)\left(\frac{\sigma_s}{\rho_s}\right)_s = 0. \tag{7.22}$$

Subsequent integration produces

$$(4\gamma - \rho)\frac{\sigma_s}{\rho_s} + \sigma + \gamma_b s = \delta(b), \tag{7.23}$$

where δ constitutes a function of integration. Accordingly, the derivatives ρ_s, ρ_b and σ_s may be expressed in terms of ρ, σ (along with γ, δ and s) by means of (7.14)$_2$, (7.16) and (7.23). Hence, the compatibility condition $\rho_{sb} = \rho_{bs}$ delivers

$$\sigma_b = \frac{2\gamma_b\sigma}{4\gamma - \rho} - 2\rho_s\frac{(\sigma + \gamma_b s)(\sigma + \gamma_b s - 2\delta) + \delta^2}{(4\gamma - \rho)^2} \tag{7.24}$$

so that the compatibility condition $\sigma_{sb} = \sigma_{bs}$ leads to

$$4\gamma_b\sigma + (\gamma_{bb}s - \delta_b)(\rho - 4\gamma) + 8\gamma_b(\gamma_b s - \delta) = 0. \tag{7.25}$$

In order for a solution to exist, this must be an admissible constraint on the system (7.14)$_2$, (7.16), (7.23), (7.24). On differentiation with respect to s and elimination of σ_s and σ by means of (7.23) and (7.25), respectively, we are led to the necessary condition

$$\rho_s = \frac{(4\gamma - \rho)[\gamma_{bb}(4\gamma - \rho) - 8\gamma_b^2]}{4\gamma_b(\gamma_b s - \delta)}, \tag{7.26}$$

which may be regarded as a Riccati equation for ρ. Its solution takes the form

$$\rho = \frac{\alpha_2(b)s^2 + \alpha_1(b)s + \alpha_0(b)}{\beta_2(b)s^2 + \beta_1(b)s + \beta_0(b)}, \tag{7.27}$$

where α_i and β_i are known functions of γ, δ and its derivatives along with a function of integration. In view of (7.25), σ is likewise rational in s and (7.14)$_2$ reduces to the polynomial form

$$\sum_{n=0}^{6} \gamma_n(b)s^n = 0. \tag{7.28}$$

It is readily shown via symbolic computation (MAPLE) that the conditions $\gamma_n = 0$ constitute an overdetermined system of ordinary differential equations for γ and δ which is inconsistent. Accordingly, the constraint (7.12) is incompatible with the Da Rios system if $\gamma \neq$ const.

Conclusion: The above analysis shows that if the symmetry (7.10), (7.11) is applied to the admissible cases discussed in the preceding then the corresponding solutions indeed constitute travelling waves and include five constants of integration. On the other hand, the complete class of travelling wave solutions admitted by the Da Rios system likewise contains five constants of integration. Hence, we arrive at the important conclusion that the conditions (7.9) do not impose any constraint on that class.

The remaining assertions rely upon results originally obtained in an investigation of the motion of an isolated vortex filament in an unbounded liquid. In that context, Da Rios [8] set down particular travelling wave solutions of the system (7.8) and discussed the motion of the associated vortex filaments wherein the variable b denotes time. In 1981, in the same physical context of the so-called 'localised induction approximation' [36], Kida [37] derived the class of rigid motions which are admitted by the Heisenberg spin equation. These motions are governed by (7.7) and correspond to the complete class of travelling wave solutions given in terms of elliptic functions of the Da Rios system. Combination of the Heisenberg spin equation $(2.3)_1$ and the constraint (7.7) produces

$$c\boldsymbol{t}_s + \boldsymbol{c} \times \boldsymbol{t} = \boldsymbol{t} \times \boldsymbol{t}_{ss}, \tag{7.29}$$

which shows that the filaments exhibit the same geometry as the centerline of symmetric elastic rods [38]. □

Remarkably, it has been shown [15,22] that the surfaces swept out by the above motions may indeed be used in the current hydrodynamics context to foliate $\mathbb{R}^3$ as dictated by the constraint $\operatorname{div} \boldsymbol{t} = 0$. In particular, nested toroidal constant pressure surfaces may be constructed if certain constants of integration are chosen appropriately. Here, it is important to note that Ω is required to vary with the foliation parameter (that is, the pressure p) unless the streamlines constitute helices. By construction, any such foliation corresponds to a multiplicity of hydrodynamic flows, which is reflected by the fact that the variation of q in the principal normal direction $\boldsymbol{n}$ is unknown. However, since the abnormality Ω is independent of s, combination of the relations (4.5) and (5.1) reveals that the velocity magnitude q is constant on individual constant pressure surfaces, that is

$$\frac{\delta q}{\delta s} = 0, \qquad \frac{\delta q}{\delta b} = 0. \tag{7.30}$$

The latter pair may be supplemented consistently by the constraint

$$\frac{\delta q}{\delta n} = \kappa q. \tag{7.31}$$

Indeed, the corresponding compatibility conditions may be shown to be satisfied on use of the commutator relations (3.4). Accordingly, the constraint (7.31) is admissible and selects a particular member of the above-mentioned class of flows which share the same geometry. In fact, this particular flow constitutes a Beltrami flow [32] in that the vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{q}$

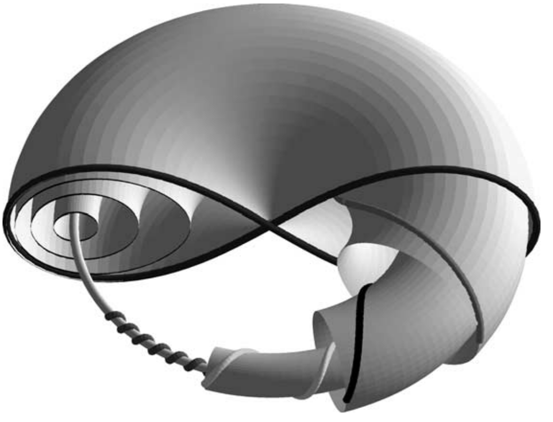

Fig. 1. The streamlines on nested toroidal constant pressure surfaces.

of the flow is parallel to $\boldsymbol{q}$. Indeed, the Masotti–Emde–Bjørgum relation (3.3)$_3$ yields

$$\boldsymbol{\omega} = \Omega \boldsymbol{q}. \tag{7.32}$$

In the case of nested tori, the streamlines wrap around the constant pressure surfaces and may be closed. However, in general, the constant pressure surfaces are 'ergodic' in that they are covered by single streamlines. A configuration of streamlines on nested toroidal constant pressure surfaces is displayed in Fig. 1.

In the mathematically equivalent context of magnetohydrostatics, this configuration has been discussed in detail in [15]. Remarkably, it coincides with that obtained by Palumbo [40] in connection with isodynamic magnetohydrostatic equilibria.

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